Acta Cryst. (1971). B27, 2036

Weighting of single isomorphous replacement Fouriers.

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(*Received* 22 *April* 1971)

If an electron density map is calculated, using as the phase angle, $\alpha_{\mathbf{h}}$, the average value of the pair of angles ($\alpha_{\mathbf{h}}+\delta_{\mathbf{h}}$ and $\alpha_{\mathbf{h}}-\delta_{\mathbf{h}}$) derived from a single isomorphous replacement procedure, then an upper bound in the error of the resulting map, derived from the 'Cauchy inequality', will be minimized when the coefficient, $|F_{\mathbf{h}}|$, in the Fourier summation is wieghted by the cosine of $\delta_{\mathbf{h}}$.

The problems of using a single pair of isomorphous crystals in a structure determination have been discussed by Kartha (1961) and Blow & Rossmann (1961). The use of a single pair of isomorphous crystals results in an ambiguous pair of phase angles which may be represented as $\alpha_{\mathbf{h}}+\delta_{\mathbf{h}}$ and $\alpha_{\mathbf{h}}-\delta_{\mathbf{h}}$; only the sign is unknown, and $\delta_{\mathbf{h}}$ is restricted to the range 0 to $\frac{1}{4}$. Both papers cited above suggest the use of $|F(\mathbf{h})|\cos 2\pi\delta_{\mathbf{h}}$ as the coefficients and $\alpha_{\mathbf{h}}$ as the phase angles in the Fourier summation as a means of circumventing the problem of choosing the sign of $\delta_{\mathbf{h}}$. This note lends theoretical support to this suggestion.

The true electron density is given by the expression:

$$\rho(\mathbf{r})=\frac{1}{V}\sum_{\mathbf{h}}|F(\mathbf{h})|\exp[2\pi i(\alpha_{\mathbf{h}}\pm\delta_{\mathbf{h}})]\exp(-2\pi i\mathbf{h}\,.\,\mathbf{r})\,,\tag{1}$$

where the correct sign for $\delta_{\mathbf{h}}$ should be chosen for each term of the summation, but is, by the nature of the procedure, not known. To stress this fact we carry the ± indication until it disappears naturally near the final step of the analysis.

Assuming a phase angle equal to $\alpha_{\mathbf{h}}$ (average of the ambiguous pair), an approximate electron density may be defined by:

$$\rho'(\mathbf{r})=\frac{1}{V}\sum_{\mathbf{h}}w_{\mathbf{h}}|F(\mathbf{h})|\exp(2\pi i\alpha_{\mathbf{h}})\exp(-2\pi i\mathbf{h}\,.\,\mathbf{r})\,,\tag{2}$$

where $w_{\mathbf{h}}$ is a weighting factor, assumed to be real, and is chosen in a manner described below to limit the difference between $\rho(\mathbf{r})$ and $\rho'(\mathbf{r})$. Although it may not be possible, through the use of a simple weighting factor, to minimize the difference, it is possible to define a least upper bound.

Subtracting equation (2) from equation (1) yields:

$$\rho(\mathbf{r})-\rho'(\mathbf{r})=\frac{1}{V}\sum_{\mathbf{h}}|F(\mathbf{h})|\exp(2\pi i\alpha_{\mathbf{h}})\exp(-2\pi i\mathbf{h}\,.\,\mathbf{r})\,[\exp(\pm 2\pi i\delta_{\mathbf{h}})-w_{\mathbf{h}}]\,.\tag{3}$$

Cauchy's (1897) inequality may be cast in the following form:

$$\left|\sum_{\mathbf{r}}a_{\mathbf{r}}b_{\mathbf{r}}\right|^2\le\sum_{\mathbf{r}}|a_{\mathbf{r}}|^2\,.\,\sum_{\mathbf{r}}|b_{\mathbf{r}}|^2\,.$$

Applying this inequality to equation (3) yields:

$$|\rho(\mathbf{r})-\rho'(\mathbf{r})|^2\le\frac{1}{V^2}\sum_{\mathbf{h}}\Big||F(\mathbf{h})|\exp(2\pi i\alpha_{\mathbf{h}})\exp(-2\pi i\mathbf{h}\,.\,\mathbf{r})\Big|^2\times\sum_{\mathbf{h}}|\exp(\pm 2\pi i\delta_{\mathbf{h}})-w_{\mathbf{h}}|^2\,.\tag{4}$$

The first summation contains only known quantities and is, therefore, fixed. All terms, $t_{\mathbf{h}}$, of the second summation are nonnegative real numbers:

$$t_{\mathbf{h}}=|\exp(\pm 2\pi i\delta_{\mathbf{h}})-w_{\mathbf{h}}|^2\,.$$

To establish a least upper bound for the squared error in the electron density, $|\rho(\mathbf{r})-\rho'(\mathbf{r})|^2$, it is only necessary to minimize each term, $t_{\mathbf{h}}$. Expanding $t_{\mathbf{h}}$ gives:

$$t_{\mathbf{h}}=1+w_{\mathbf{h}}^2-2w_{\mathbf{h}}\cos 2\pi\delta_{\mathbf{h}}\,.$$

Note that this expansion involves the product of a number and its complex conjugate, leading to the disappearance of the sign ambiguity. To minimize $t_{\mathbf{h}}$, we differentiate with respect to $w_{\mathbf{h}}$ and set the result equal to zero, yielding:

$$\frac{\partial t_{\mathbf{h}}}{\partial w_{\mathbf{h}}}=2w_{\mathbf{h}}-2\cos 2\pi\delta_{\mathbf{h}}=0\,,$$

or

$$w_{\mathbf{h}}=\cos 2\pi\delta_{\mathbf{h}}\,.$$

The use of this weighting factor, therefore, reduces the maximum error in the approximate electron density [equation (2)] to its least value. Blow & Crick (1959) arrived at a similar result by a somewhat different route.

Helpful discussions with Professor Jurg Waser and Dr Richard E. Marsh during the preparation of this manuscript are gratefully acknowledged.

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* Contribution No. 4242 from the Gates and Crellin Laboratories of Chemistry. This investigation was supported in part by Research Grant GB-6617 from the National Science Foundation, and in part by Public Health Service Research Grant No. 12121 from the National Institute of General Medical Sciences.